REGULAR p-GROUPS. II

BY

AVINOAM MANN

ABSTRACT

We investigate minimal irregular p-groups, and derive several criteria for regularity as a consequence. For example, a p-group is regular if all its subgroups have "few" generators. It is also shown that for varieties, regularity is equivalent to "good" power structure. We close with some examples.

In this note all groups are finite p-groups. In [8] (henceforth to be referred to as I) we have defined such a group P to be *minimal irregular* if P is irregular, but all proper sections of P are regular p-groups. Then every irregular group involves some minimal irregular group; therefore, investigations of minimal irregular groups yield criteria for regularity. Continuing I, we further investigate minimal irregular p-groups, and derive some criteria for regularity. Some of these criteria are:

If the (finite p) group G has "many" regular factor groups, then G is regular (see Proposition 2).

If all subgroups of G' can be generated by $\frac{1}{2}(p-1)$ elements, then G is regular (Proposition 3).

If in G, and also in all groups in $\operatorname{Var} G$, products of elements of order p again have order p (or products of p'th powers are p'th powers) then G is regular (Theorem 7).

Minimal irregular groups were investigated also by P. M. Weichsel [10] and J. R. J. Groves [5]. In particular, it is proved in [5] that such a group has class $c(p)$ at most (improving the bound $c(p)+1$ of I), where $c(p)$ is the maximal class possible for a 2-generator finite group of exponent p. In Section 4 we give some

Received October 30, 1972

examples of minimal irregular groups which show that such groups exist of all classes between p and $c(p)$.

Notation and terminology follow I, familiarity with which is assumed. In addition, we use $\operatorname{Var} G$ for the variety generated by the group G, C_n for the cyclic group of order n, and call a group G V-regular if all finite groups in $\operatorname{Var} G$ are regular. Recall also that $c(p)$ was defined in the previous paragraph.

1.

The first criterion is just a better formulation of Corollary 1 of I.

PROPOSITION 1. *Let G be a finite p-group of exponent p^e. Then G is regular if and only if all sections of exponent p^2 of $G \times C_{p^e}$ are regular.*

PROOF. Certainly if G is regular so are $G \times C_{pe}$ and all its sections. Assume G is irregular. Then G has a minimal irregular section P, say, of exponent p^f, $f \leqq e$. Let $P_i, i=1,2$, be a group isomorphic to P under an isomorphism $x \leftrightarrow x_i$ ($x \in P$, $x_i \in P_i$) By the proof of Theorem 2 (h) of I, $H/\mho_2(H)$ is irregular, where

$$H = \langle (a_1, a_2), (b_1, u_2) \rangle \subseteq P_1 \times P_2$$

a, b are appropriate generators of P, and u is an arbitrary element of P'. Choosing $u = 1$, we have $H \subseteq P_1 \times \langle a_2 \rangle$; $|\langle a_2 \rangle| = p^f \Rightarrow \langle a_2 \rangle \subseteq C_{p^e}$ so $H/\mho_2(H)$ is an irregular section of exponent p^2 of $G \times C_{p^e}$.

The next result shows that the class of regular groups possesses a closure property similar to (but weaker than) subdirect closure (which regular groups do not enjoy).

PROPOSITION 2. *Let G be a finite p-group. Assume G contains a central elementary abelian subgroup N of order p^2, such that G/M is regular for all $\{1\} \neq M \subseteq N$ Then G is regular.*

PROOF. We use induction on $|G|$. Let H be a proper subgroup of G. If $H \supseteq N$ then H is regular by induction. If $H \not\supseteq N$, there exists an $M \subseteq N$ with $|M| = p$ such that $H \not\supseteq M$, and then $H \cong HM/M \subseteq G/M$ is regular. Next, let $\{1\} \neq K \lhd G$ If $K \cap N \neq \{1\}$, then G/K is regular, being an image of $G/K \cap N$. If $K \cap N = \{1\}$, then G/K and NK/K satisfy the assumptions of the proposition. Thus all proper sections of G are regular. If G is irregular, it is minimal irregular, which is impossible since G contains more than one minimal normal subgroup.

PROPOSITION 3. *Let G be a finite p-group. If all subgroups of G' can be generated by $\frac{1}{2}(p-1)$ elements, then G is regular.*

PROOF. By induction, we may assume that G is a minimal irregular group. Let $\operatorname{cl} G = c$, and let $H = G_{c+1-1/2(p+1)}$. Then H is abelian of order at least $p^{1/2(p+1)}$. Since G_2 is of exponent p, H is elementary abelian, and so H has a least $1/2(p+1)$ generators, a contradiction.

COROLLARY 4. *Let G be a finite p-group, for $p \geqq 5$. If G' can be generated by 2 elements, then G is regular.*

Indeed, a result of Blackburn [1, Th. 1] shows that in this case, all subgroup of G' can be generated by 2 elements, and Proposition 3 applies. (This corollary follows also from Hall's criterion [7, III. 10. 13]).

For 3-groups, Proposition 3 reduces to the well-known result that groups with cyclic commutator subgroups are regular. Moreover, any 3-group of maximal class and order at least 3^5 has all its subgroups generated by two elements, but is irregular [7, III. 14. 17 and 14. 22].

In I, we have shown that a finite p-group G is regular if, for all 2-generator subgroups H, $\operatorname{cl} H/\mho_1(H) \leqq p-2$. (In Corollary 2 and Theorem 2(c) of I, it is misprinted Ω, instead of $\mho$). An alternative formulation for this criterion is:

A finite p-group G is regular if all 2-generator sections of G of exponent p have class $p-2$ at most.

Still another way of writing this condition is

$$H_{p-1} \subseteq \mho_1(H) \tag{A}$$

where H varies over all 2-generator subgroups of G. Inclusion (A) is formally similar to the inclusion (H as before)

$$H_p \subseteq \mho_1(H_2) \tag{B}$$

which also implies regularity. Indeed, assuming (A) and knowing that G is regular, we have, by [7, III. 10. 8 (a)]

$$H_p = [H_{p-1}, H] \subseteq [\mho_1(H), H] = \mho_1(H_2)$$

so (A) implies (B). We now consider conditions "dual" to (A) and (B).

PROPOSITION 5. *The finite p-group G is regular, if it satisfies either* (C) *or* (D) *below, where H varies over all 2-generator sections of G.*

$$\Omega_1(H) \subseteq Z_{p-2}(H) \tag{C}$$

$$\Omega_1(H/Z(H)) \subseteq Z_{p-1}(H). \tag{D}$$

PROOF. Condition (C), of course, implies both (A) and (D), so we consider only the latter. Now if P is a minimal irregular p-group, then $P = \Omega_1(P \bmod Z(P))$, $P \neq Z_{p-1}(P)$ so G does not involve P, and hence G is regular.

2.

Let P be a minimal irregular group. By I, P satisfies the law

$$(1) \qquad (x[y,z])^p = x^p$$

which implies (e.g. by taking $x = y$)

$$(2) \qquad [x^p, z] = 1.$$

Let F be the free group on two generators, a and b, in Var P.

Let $Z = \langle a^p, b^p \rangle \subseteq Z(F)$ and let $p^e = \exp P = \exp F$. Then Z is a 2-generator abelian group and $\exp Z = p^{e-1}$. In the epimorphism

$$F \to F/F' \cong C_{p^e} \times C_{p^e}$$

Z is mapped onto $C_{p^{e-1}} \times C_{p^{e-1}}$, which implies

$$Z \cong C_{p^{e-1}} \times C_{p^{e-1}}, \; Z \cap F' = 1.$$

If $\mho_1(F) \cap F' = \{1\}$, then F is a subdirect product of a group of exponent p and an abelian group, and F is regular. Thus $Z \neq \mho_1(F)$. Now we have

$$\Phi(F) = \mho_1(F)F' = ZF', \; \mho_1(F) = Z \times (\mho_1(F) \cap F')$$

$$\mho_1(F) \subseteq Z(F) \subseteq \Phi(F), \; Z(F) = Z \times (Z(F) \cap F').$$

(If $Z(F) \nsubseteq \Phi(F)$, we can find two generators for F, one of which is in $Z(F)$, so F is abelian.) By (1) F' has exponent p, so $\mho_1(F) \cap F'$ and $Z(F) \cap F'$ are elementary abelian. Also

$$F' \subseteq \Omega_1(F) = F' \times \Omega_1(Z).$$

We have proved

PROPOSITON 6. *$\mho_1(F)$ and $Z(F)$ are direct products of at least three cyclic subgroups, of which two have order p^{e-1} and the others order p. Also $F/\Omega_1(F) \cong C_{p^{e-1}} \times C_{p^{e-1}}$.*

Regular groups have "nicely behaved" power structure. This, however, does not characterize regular groups. Thus direct products of regular groups, or weakly regular groups, have similar properties. We show now that on the level of varieties, "good" power structure is equivalent to regularity.

THEOREM 7. *Let G be a finite p-group. Then the following are equivalent:*

a. *G is V-regular.*

b. *For all* $H \in \operatorname{Var} G$, $\Omega_1(H)$ *contains only elements of order* p (or 1).

c. *For all* $H \in \operatorname{Var} G$, $\mho_1(H)$ *contains only p'th powers.*

d. *For all finite* $H \in \operatorname{Var} G$, $|\mho_1(H)| = |H:\Omega_1(H)|$.

PROOF. If G is V-regular, (b), (c) and (d) are all well-known properties of regular groups. Assume G is not V-regular. Then, for each of the statements (b) (c), (d), we have to find an $H \in \operatorname{Var} G$ which violates this statement.

$\operatorname{Var} G$ contains some minimal irregular group P, hence some free group F of the type discussed in Proposition 6. By this proposition, F violates (d).

F/Z is generated by aZ and bZ, which have order p. Since $\mho_1(F) \nsubseteq Z$, F/Z does not have exponent p. Thus F/Z violates (b).

Assume that F satisfies (c). Then $a^p b^p = c^p$ for some $c \in F$. Write $c = \omega(a,b)$ as a word in a, b. Then $\operatorname{Var} F = \operatorname{Var} P$ satisfies the law

$$x^p y^p = (\omega(x, y))^p. \tag{3}$$

Write $\omega(a,b) = a^i b^j u$, where $u \in F'$. Then (3) and (1) imply

$$x^p y^p = (x^i y^j)^p. \tag{4}$$

Let $\exp F = \exp \operatorname{Var} F = p^e$, and substitute in (4) $x = 1$ or $y = 1$. We get

$$x^p = x^{ip} \Rightarrow i = 1 + kp^{e-1}, \ y^p = y^{jp} \Rightarrow j = 1 + lp^{e-1}.$$

Then (2) and (4) yield

$$x^p y^p = (xy\, x^{kp^{e-1}} y^{p^{e-1}})^p = (xy)^p \tag{5}$$

so F is p-abelian and regular, a contradiction. Therefore F violates (c).

We give two further results concerning varieties.

PROPOSITION 8. *Let G be a 2-generator finite p-group satisfying* (1). *Then every proper subgroup of G is p-abelian* (*hence regular*).

PROOF. We have $|G: \Phi(G)| = p^2$. Let H be a maximal subgroup of G, and let $a \in H$ be such that $H = \langle a, \Phi(G) \rangle$. Let $b, c \in H$. Then

$$b = a^i u_1 v_1, \ c = a^j u_2 v_2 \,; \ u_1, u_2 \in \mho_1(G) \,; \ v_1, v_2 \in G'.$$

Notice that G satisfies (2), so $u_1, u_2 \in Z(G)$. Then

$$(bc)^p = (a^{i+j} u_1 u_2 v_1^{a^j} v_2)^p = a^{(i+j)p} u_1{}^p u_2{}^p$$
$$= a^{ip} u_1{}^p a^{jp} u_2{}^p = b^p c^p.$$

PROPOSITION 9. *Let P be a minimal irregular group. Then* Var P *can be generated by a cyclic group and a minimal irregular group of exponent* p^2.

PROOF. Let $\exp P = p^e$. We construct H as in Proposition 1, so that we have (where $C = C_{p^e}$)

$$H \subseteq P \times C, \ H/\mho_2(H) \text{ is irregular.}$$

By its construction, H is a subdirect product of P and C, so

$$\mathrm{Var}\,H = \mathrm{Var}(P, C) = \mathrm{Var}\,P.$$

Choose $N \lhd H$ maximal such that $N \supseteq \mho_2(H)$ and H/N is irregular. By Proposition 8, H/N is a minimal irregular group. We consider H, N and P as subgroups of $P \times C$. Assume $N \cap P \neq \{1\}$. Then $N \cap P \supseteq T$, T being the unique minimal normal subgroup of P, since $N \cap P \lhd P$. But then

$$H/N \cong H/T \big/ N/T, \ H/T \subseteq (P \times C)/T \cong P/T \times C$$

and $P/T \times C$ is regular. Thus

$$N \cap (H \cap P) = N \cap P = \{1\}.$$

Also, $H/H \cap P \cong C$, and so

$$\mathrm{Var}\,P = \mathrm{Var}\,H = \mathrm{Var}(H/N, C).$$

3.

DEFINITION. Let Q be a finite p-group. Q is an *s-minimal irregular group* if Q is irregular, but all its proper subgroups are regular.

If P is minimal irregular, and F is free of rank 2 in Var P, then by Proposition 8 F is s-minimal irregular. By Proposition 6, F has more than one minimal normal subgroup, so F is not minimal irregular.

PROPOSITION 10. *Let Q be an s-minimal irregular group. Then Q has two generators, and $Q/\mho_1(Q')$ is also s-minimal irregular.*

PROOF. An irregular group must have a 2-generator irregular subgroup, hence Q has two generators. Let $a, b \in Q$. If $\langle a, b \rangle \neq Q$ then

$$(ab)^p = a^p b^p c^p, \ c \in \langle a, b \rangle'$$

since $\langle a, b \rangle$ is regular. Assume that $Q/\mho_1(Q')$ is regular, and that $\langle a, b \rangle = Q$. Then

$$(ab)^p = a^p b^p c^p d, \ c \in Q', \ d \in \mho_1(Q').$$

Because Q' is regular, we get

$$c^p d \in \mho_1(Q'),\ (ab)^p = a^p b^p e^p,\ e \in Q' = \langle a, b \rangle'$$

and Q is regular. Therefore $Q/\mho_1(Q')$ is irregular and, with Q, it has all proper subgroups regular. Notice that $Q/\mho_1(Q')$ has a derived group of exponent p.

PROPOSITION 11. *Let Q be an s-minimal irregular p-group. Assume that* $\exp Q' = p$, *and let* $\mathrm{cl}\, Q = c$. *Then*

a. *Q can be generated by two elements.*

b. *Q satisfies* (1) *and* (2).

c. $\mho_1(Q) \subseteq Z(Q) \subseteq \Phi(Q)$; $Z_{c-1}(Q) = \Phi(Q)$.

d. *All proper subgroups of Q have class less than c.*

e. *Q is p^2-abelian.*

f. $c \leqq c(p)$.

g. *A* 2*-generator group in* $\mathrm{Var}\, Q$ *is either regular or s-minimal irregular.*

Most of these statements are proved in Theorem 2 of I. Proposition 8 implies (g), while (f) follows from (1) and Lemma 2.1 of Groves [5].

We note that (d), for minimal irregular groups, follows also from [6, Corollary]. We give now a slightly sharper result, which implies (d) in all cases.

PROPOSITION 12. *Let G be a* 2*-generator finite p-group of class c. Then*

a. $Z_{c-1}(G) \subseteq \Phi(G)$.

b. $\exp G/Z_{c-1}(G) = \exp G_c$.

c. *If $Z_{c-1}(G) = \Phi(G)$, all proper subgroups of G have class less than c.*

d. *If $H \subset G$, then* $H_c \subseteq \mho_1(G_c)$.

PROOF. If $Z_{c-1}(G) \nsubseteq \Phi(G)$, we can find generators, a and b, for G with $a \in Z_{c-1}(G)$. Then all c-fold commutators in a and b are 1, and $\mathrm{cl}\, G < c$.

Since $\mathrm{cl}\, G = c$, we have, for $a_1, \cdots, a_c \in G$, $[a_1, a_2, \cdots, a_c]^{p^e} = [a_1^{p^e}, a_2, \cdots, a_c]$ which implies (b), while (c) is proved as in Theorem 2(e) of I. By (b), $Z_{c-1}(G) = \Phi(G)$, which is equivalent to $\mho_1(G) \subseteq Z_{c-1}(G)$, is equivalent also to $\exp G_c = p$. Applying this fact and (c) to $G/\mho_1(G_c)$, one gets (d).

A corollary of (d) is that 2-generator p-groups are s-critical, since all proper subgroups satisfy the law $[x_1, \cdots, x_c]^{p^{e-1}} = 1$ ($p^e = \exp G_c$) which G does not satisfy. This fact was already noted by P. M. Weichsel [9, proof of 4.1], and it follows also from [4, Corollary 2].

4.

Let $B(p)$ be the largest 2-generator finite group of exponent p, and let $c(p)$ be its class. For a given $p \leqq c \leqq c(p)$, let $B = B(p,c)$ be $B(p)/B(p)_{c+1}$. Let D be the wreath product of two cyclic groups of order p. Let $B(p,c) = \langle a,b \rangle$ and $D = \langle c,d \rangle$, where c and d have order p. Let

$$H = \langle (a,c),\ (b,d) \rangle \subseteq B \times D.$$

Then H is a subdirect product of B and D. H is of class c, irregular (with D), and s-minimal irregular, by Proposition 8. Let $N = \mho_1(H)$, then

$$N \neq \{1\},\ N \subseteq \mho_1(B \times D) = \mho_1(D) \cong C_p \Rightarrow |N| = p.$$

As H is generated by elements of order p, any regular factor group of H is of exponent p, so for $K \lhd H$:

(E) $$H/K \text{ is regular} \Leftrightarrow K \supseteq N.$$

Now for all $L \lhd H$, H/L has all proper subgroups regular, so H/L is a minimal irregular group if and only if H/L is irregular, but all proper factor groups of H/L are regular. By (E),

(F) $$H/L \text{ is minimal irregular} \Leftrightarrow L \text{ is maximal w.r.t. } L \cap N = \{1\}.$$

The next lemma will show that at least for one such L, $\mathrm{cl}\, H/L = c$, and so there exist minimal irregular groups of all classes c for which $p \leqq c \leqq c(p)$.

LEMMA 13. *Let G be a finite p-group, and N a minimal normal subgroup. Let $\mathscr{L} = \{L \lhd G \mid L$ is maximal w.r.t. $L \cap N = \{1\}\}$. Then the intersection of all members of $\mathscr{L}$ is trivial.*

PROOF. We use induction on $|G|$. First, if N is the unique minimal normal subgroup of G, then $\mathscr{L} = \{1\}$ and the result is trivial. So let M be another minimal normal subgroup of G. Let $\mathscr{L}(M) = \{L \in \mathscr{L} \mid L \supseteq M\}$. Then $\mathscr{L}(M) = \{L \lhd G \mid L/M$ is maximal in G/M w.r.t. $L/M \cap NM/M = \{1\}\}$ so by induction

$$M = \cap L(L \in \mathscr{L}(M)).$$

But NM contains also a minimal normal subgroup, M_1 say, distinct from both N and M. Then we find that also M_1 is the intersection of certain members of $\mathscr{L}$, and so

$$\cap L(L \in \mathscr{L}) \subseteq M \cap M_1 = \{1\}.$$

If we replace D by any other minimal irregular group, the preceding construction still yields minimal irregular groups, provided we replace $\mho_1(B\times D)$ by the verbal subgroup corresponding to the word $(xy)^p y^{-p} x^{-p}$. Moreover, by [5, 2.2], if we let D vary over all minimal irregular groups which are metabelian and (hence) of class p, our construction yields all minimal irregular groups.

Groves' result just referred to focuses attention on minimal irregular groups which are metabelian, of class p and exponent p^2. We now describe a construction for such groups.

Let G be the group

$$\langle a,b,c_{ij}\ (i,j=1,2,\cdots,p-1)\ a^p = b^p = c_{ij}^p = 1, [b,a] = c_{11},$$

$$[c_{ij},a] = c_{i,j+1}(j = 1,\ \cdot,p-2),\ [c_{ij},b] = c_{i+1,j}(i = 1,\cdots,p-2),$$

$$[c_{i,p-1},a] = 1,\ [c_{p-1,j}b] = 1,\ [c_{ij},c_{kl}] = 1\rangle.$$

By [2, 2.1]

$$|G| = p^{2+(p-1)^2},\ |G'| = p^{(p-1)^2},\ G_k = \langle c_{ij} \mid i+j \geqq k\rangle.$$

For $1 \leqq k \leqq p-1$, let

$$N_k = \langle c_{ij} \mid i > k \text{ or } i > p-k\rangle.$$

Then $N_k \lhd G$, and $P = G/N_k$ satisfies

$$P_p = Z(P) = \langle c_{k,p-k}N_k\rangle,\ P_{p+1} = 1.$$

P is irregular, since regular metabelian p-groups with 2-generators and derived groups of exponent p have class less than p (e.g. [3]). Since P_p is the unique minimal normal subgroup of P, all proper factor groups of P have class less than p, and so are all proper subgroups of P, by Proposition 12. Thus, P is a minimal irregular group, of order $p^{2+k(p-k)}$. In particular, a metabelian minimal irregular p-group of exponent p^2 is not necessarily of maximal class.

References

1. N. Blackburn, *On prime-power groups in which the derived group has two generators*, Proc. Cambridge Philos. Soc. **53** (1957), 19–27.

2. N. Blackburn, *On prime-power groups with two generators*, Proc. Cambridge Philos. Soc. **54** (1958), 327–337.

3. W. Brisley and I. D. MacDonald, *Two classes of metabelian p-groups*, Math. Z. **112** (1969), 5–12.

4. R. M. Bryant, *Finite splitting groups in varieties of groups*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 169–172.

5. J. R. J. Groves, *On minimal irregular p-groups*, J. Austral. Math. Soc., to appear.

6. C. Hobby and C.R.B. Wright, *A generalization of a theorem of N. Ito on p-groups*, Proc. Amer. Math. Soc. **11** (1960), 707–709; Errata, **12** (1961), 1000.

7. B. Huppert, *Endliche Gruppen I*, Grund. Math. **134**, Berlin, 1967.

8. A. Mann, *Regular p-groups*, Israel J. Math. **10** (1971), 471–477.

9. P. M. Weichsel, *On critical p-groups*, Proc. London Math. Soc. (3) **14** (1964), 83–100.

10. P. M. Weichsel, *Just-irregular p-groups*, Israel J. Math. **10** (1971), 359–363.

The Hebrew University of Jerusalem
Jerusalem, Israel